Review For Exam 2

The directions for the exam are as follows:

"WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. Note that you must do at least 10 problems correctly to get 100. Write neatly and legibly in the space provided. SHOW YOUR WORK!"

1. In other words, the exam consists of 10 core problems and 3 extra-credit problems. If you wish, you can do all the 13 problems, but your score will only add up to 100 points. Partial credit will be given.

2. Focus on problems indicated in red.

3. Also remember that you are allowed to use a scientific calculator.

Section 2.3

1. Go over problems 1-30 on HW # 9 **(1-5, 9-12, 15-16, 20-27)**

2. Calculate the partial derivatives for the following functions:

   (a) $f(x,y,z) = x^y$ [Hint: $x^y = e^{y\ln(x)}$]

   (b) $f(x,y) = \sin(x sin(y))$

   (c) $f(x,y,z) = \sin(x\sin(y\sin(z)))$

   (d) $f(x,y,z) = x^{y^z}$ [Hint: refer to part (a)]

   (e) $f(x,y,z) = x^{y+z}$

   (f) $f(x,y,z) = (x+y)^z$

   (g) $f(x,y) = \sin(xy)$

   (h) $f(x,y) = [\sin(xy)]^{\cos(3)}$

3. Find the partial derivatives of the following functions (where $g: \mathbb{R} \to \mathbb{R}$ is continuous):

   (a) $f(x,y) = \int_a^{x+y} g$ [Hint: Use the fundamental theorem of calc.]

   (b) $f(x,y) = \int_y^x g$

(c) $f(x,y)=\int_a^{xy} g$

(d) $f(x,y)=\int_a^{\int_b^y g} g$

4. If $f(x,y)=x^{x^{x^{xy}}}+\ln(x)\left(tan^{-1}\left(tan^{-1}\left(tan^{-1}(\sin(\cos xy)-\ln(x+y)\right.\right.\right.$,

find $\frac{\partial f}{\partial y}(1,y)$.   [Hint: There is an easy way to do this.]

5. Find the partial derivatives of f in terms of the derivatives of $g,h\colon \mathbb{R}\to\mathbb{R}$.
(a) $f(x,y)=g(x)h(y)$

(b) $f(x,y)=g(x)^{h(y)}$

(c) $f(x,y)=g(x)$

(d) $f(x,y)=g(y)$

(e) $f(x,y)=g(x+y)$

6. Given a function $f\colon \mathbb{R}^2\to\mathbb{R}$, what are the conditions for which the mixed partials $D_{1,2}f(a,b)$ and $D_{2,1}f(a,b)$ are equal at the point (a,b)? (i.e. what conditions on the mixed partials are enough to insure that $\frac{\partial^2 f}{\partial x\partial y}(a,b)=\frac{\partial^2 f}{\partial y\partial x}(a,b)$?)

7. **(Possible Extra-Credit)** Define $f\colon \mathbb{R}^2\to\mathbb{R}$ by
$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

(a) Show that $\frac{\partial f}{\partial y}(x,0)=x$ for all x and $\frac{\partial f}{\partial x}(0,y)=-y$ for all y.

(b) Show that $\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)$

8. Explain the difference between our concept of derivative in single-variable calculus versus multi-variable calculus.

9. Let $f(x)=\sin(x)$. Calculate:

(a) $f'(\pi/2)$

(b) $Df(\pi/2)$

10. Calculate the total derivative of f:

(a) $f(x, y, z) = x^y$ at the point (a, b, c)

(b) $f(x, y, z) = (x^y, z)$ at the point (a, b, c)

(c) $f(x, y) = \sin(x\, sin(y))$ at the point (a, b)

(d) $f(x, y, z) = \sin(x\, sin(y\, sin(z)))$ at the point (a, b, c)

(e) $f(x, y, z) = x^{y^z}$ at the point (a, b, c)

(f) $f(x, y, z) = x^{y+z}$ at the point (a, b, c)

(g) $f(x, y, z) = (x + y)^z$ at the point (a, b, c)

(h) $f(x, y) = \sin(xy)$ at the point (a, b)

(i) $f(x, y) = [\sin(xy)]^{\cos(3)}$ at the point (a, b)

(j) $f(x, y) = (\sin(xy), \sin(x\, sin(y)), x^y)$ at the point (a, b).

11. Find the total derivative of f (where $g: \mathbb{R} \to \mathbb{R}$ is continuous):

(a) $f(x, y) = \int_a^{x+y} g$ at the point (h, k).

(b) $f(x, y) = \int_a^{xy} g$ at the point (h, k).

(c) $f(x, y, z) = \int_{xy}^{\sin(x \sin(y \sin(z)))} g$

12. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. What is the relationship between $Df(\vec{a})$ and f? ■

13. Use differential approximation to estimate $\sqrt{8.9} + \sqrt[3]{8.1}$ ■

14. Find the equation of the tangent plane to the surface

(a) $z = x^2 + (x + 1)y^2$ at the point $(1, -2, 9)$

(b) $z = 2x - 5y - 1$ at the point $(0, 1, -6)$

15. Suppose that $f(2,-5) = -1$ and $Df(2,-5)(x,y) = x + 4y$. Estimate the value of $f(2.1,-4.9)$.

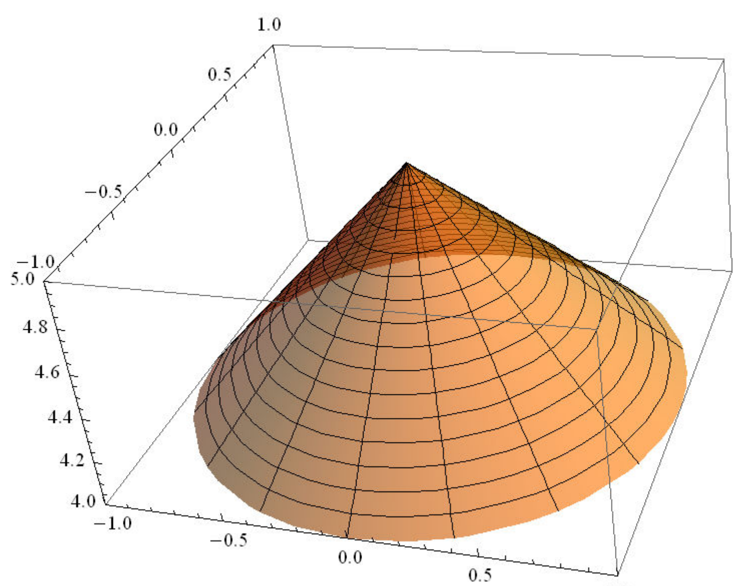
16. **(Possible Extra-Credit)** Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Show that f is differentiable at $x = a$ (in the calc. I sense) if and only if there exists a linear function $T: \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to a} \frac{f(x)-f(a)-T(x-a)}{|x-a|} = 0$.

17. **(Possible Extra-Credit)** A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is said to be differentiable at $\vec{x} = \vec{a}$ if there exists a linear function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})-f(\vec{a})-T(\overline{x-a})}{||\overline{x-a}||} = \vec{0}$. Show that if such T exists, then it must be unique. (Hence the notation $T = Df(\vec{a})$ is justified)

18. **(Possible Extra-Credit)** Show that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x} = \vec{a}$ then it must be continuous at $\vec{x} = \vec{a}$.

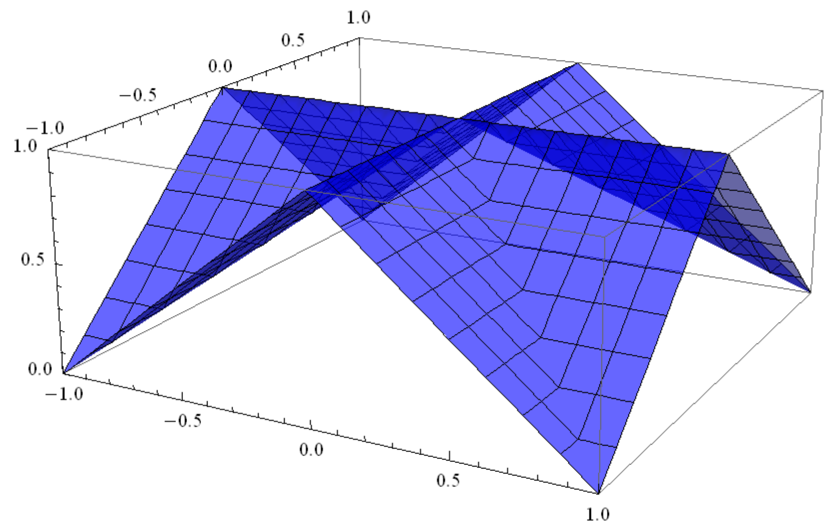
19. **(Possible Extra-Credit)** Show that if $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x} = \vec{a}$ then all the partial derivatives $\frac{\partial f}{\partial x_k}(\vec{a})$ (k = 1, 2, ..., n) exist and satisfy the equation $\frac{\partial f}{\partial x_k}(\vec{a}) = Df(\vec{a})(\overrightarrow{e_k})$.

20. **(Possible Extra-Credit)** The graph of the function $f(x,y) = 5 - \sqrt{x^2 + y^2}$ is shown below:



Without doing any computations, do you think f is differentiable at (0, 0)? Use your geometric intuition.

21. **(Possible Extra-Credit)** The "Victorian cottage roof" is the graph of the function $f(x, y) = 1 - \min\{|x|, |y|\}$ is shown below:



(a) Using your geometric intuition or using the formula of f, compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(b) Using part (a) what would be your formula for $Df(0,0)$?

(c) According to your intuition, is f differentiable at (0,0)? Is the function obtained in part (b) the derivative of f at (0, 0)?

22. **(Possible Extra-Credit)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{|x||y|}$. Show that f is not differentiable at (0,0).

23. **(Possible Extra-Credit)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

(a) Is f continuous at (0,0)? Justify your answer.

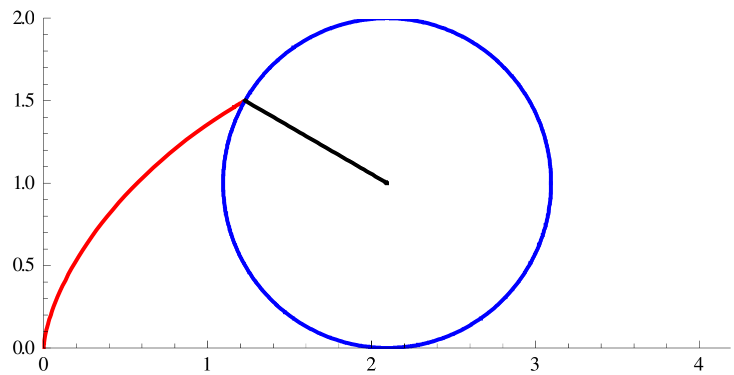
(b) Is f differentiable at (0, 0)? Justify your answer.

24. **(Possible Extra-Credit)** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(\vec{x})| \leq \|\vec{x}\|^2$. Show that f is differentiable at $\vec{0}$.

Section 2.4

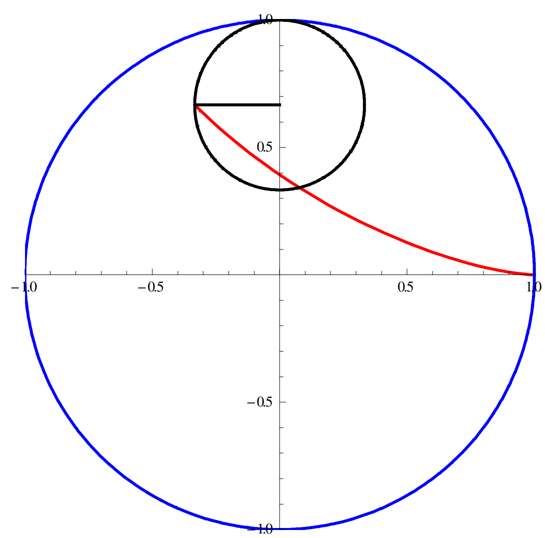
1. Go over problems 1-20 on HW # 8 **(1-5, 13-15, 17-20)**

2. A **cycloid** is a curve that is traced by a point on a rolling circle that travels without slipping along the x-axis.



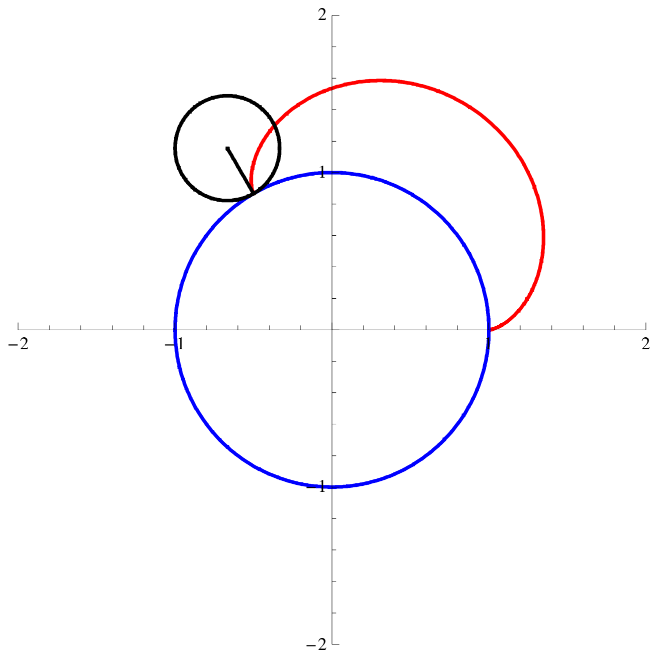
Find a path function that traces this curve. Show your work. ■

3. A **hypocycloid** is a curve traced by a point on a rolling circle of radius *r* that travels within another circle of radius *R* without slipping



Find a path function that traces this curve. Show your work. ■

4. An **epicycloid** is a curve traced by a point on a rolling circle of radius *r* that travels on the outside of another circle of radius *R* without slipping.



Find a path function that traces this curve. Show your work. ■

5. Let $p(t) = (t, \cos t, e^{2t})$.

(a) Compute $p'(0)$

(b) Compute $Dp(0)$

(c) If $p(t)$ represents the position of a particle at time *t*, what is the physical interpretation of your calculations in (a) and in (b)?

6. Calculate the curvature.

(a) $r(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$

(b) $r(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$

(c) $r(t) = (t, 3\cos t\,, 3\sin t)$

(d) $r(t) = (\sqrt{2}t, e^t, e^{-t})$

(e) $r(t) = \left(t, \frac{1}{2}t^2, t^2\right)$

(f) $r(t) = (\cos^3 t\,, \sin^3 t)$

Section 2.5

1. Go over problems 1-24 on HW # 10 **(1-3, 6-9, 14-16, 18, 21-22, 24)**

2. Let $p(r,\theta) = (r\cos\theta, r\sin\theta)$, $f(x,y) = (x, x+y, x-y)$, and $g(x,y,z) = xyz$. Compute $D(g \circ f \circ p)(1, \frac{\pi}{2})$. ■

3. **(Possible Extra-Credit)** Use chain rule to derive the expression for product rule. In particular, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{a} \in \mathbb{R}^n$, then $D(fg)(\vec{a})(\vec{x}) = g(\vec{a})Df(\vec{a})(\vec{x}) + f(\vec{a})Dg(\vec{a})(\vec{x})$.

4. **(Possible Extra-Credit)** Use chain rule to derive the expression for quotient rule. In particular, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{a} \in \mathbb{R}^n$ with $g(\vec{a}) \neq 0$, then $D(f/g)(\vec{a})(\vec{x}) = \frac{g(\vec{a})Df(\vec{a})(\vec{x}) - f(\vec{a})Dg(\vec{a})(\vec{x})}{[g(\vec{a})]^2}$.

Section 2.6

1. Go over problems 1-16 on HW # 11 **(1-3, 5-9, 12-14)**

2. **(Possible Extra-Credit)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by
$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   (a) Is f continuous at $(0,0)$?

   (b) Do all the directional derivatives $D_{\vec{u}}f(0,0)$ exist at $(0,0)$?

   (c) Is f differentiable at $(0,0)$?

   Justify all your assertions.